Fast distributed approximation in planar graphs

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Model of computation

distributed, message passing, synchronous
(known sa $LOCAL$ in D. Peleg book "Distributed Computing...")

communication graph: **planar**

Our problems

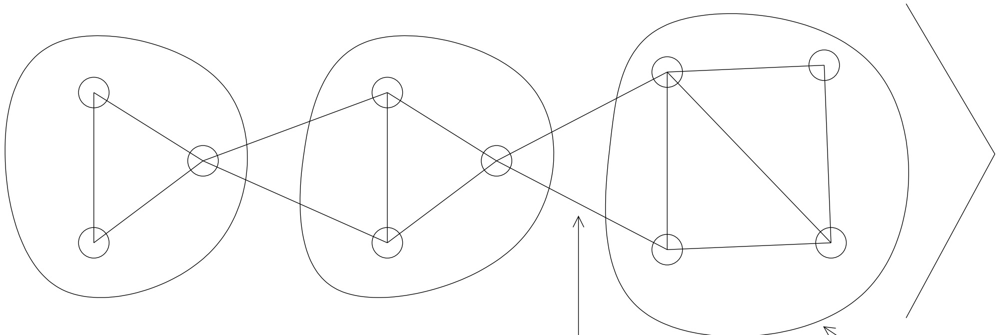
basic graph problems: MWIS, MIS, MDS

deterministic $(1 \pm \epsilon)$-approximation algorithms
in $O(\log^* n)$ time (number of rounds)

randomized $(1 \pm \epsilon)$-approximation algorithm in $O(1)$ time for MIS

some lower bound

Main tool for deterministic approx.: clusters



clusters: partiton of $V(G)$ into disjoint sets $V_1, \ldots, V_k$
each $G[V_i]$ is connected and has constant diameter

edge weight function $\omega : E(G) \rightarrow R^+$

we can compute clusters s.t.: weight of connectors $< \epsilon\ \omega(E(G))$,
for constant $\epsilon > 0$, in time $O(\log^* n)$, deterministically ...

Input: planar graph G, weights on edges ω
Output: clusters with weight of connectors $< \epsilon\, \omega(E(G))$

definition a directed graph F with max-out-degree $= 1$
is called a *pseudo forest*

1. Each vertex $v \in V(G)$ puts outgoing arrow on the heaviest $\{v, u\}$ (we get pseudo forest F s.t. $\omega(E(F)) > \frac{1}{6}\omega(E(G)))$

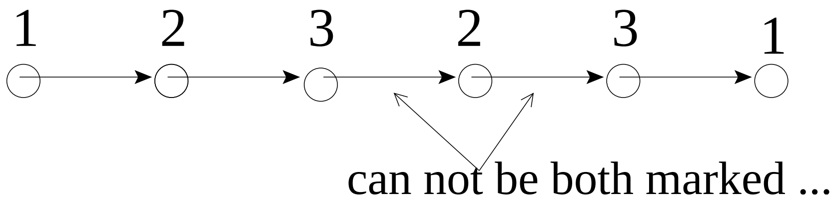
2. We compute 3-vertex-coloring of F using CV algorithm, in $O(\log^* n)$ time

3. Each vertex $v \in V(F)$, in parallel:

   (a) if $color(v) = 1$ then v **marks** all incoming edges or outgoing edge (whatever is heavier)

   (b) if $color(v) = 2$ then v **marks** all incoming edges to color 3 or outgoing edge to color 3 (whatever is heavier)

Let Q_i be a connected component of the graph induced by marked edges in F: $diam(Q_i) < 10$



4. In each Q_i find disjoint stars of weight $> \frac{1}{2}\omega(E(Q_i))$

We have disjoint stars of weight $> \frac{1}{24}\omega(E(G))$ ...

Iterate $\log(\frac{1}{\epsilon}) / \log(\frac{24}{23})$ times:

1. Call the above procedure

2. Contract stars to vertices and recompute weights of edges (must be = the sum of weight of edges between stars)

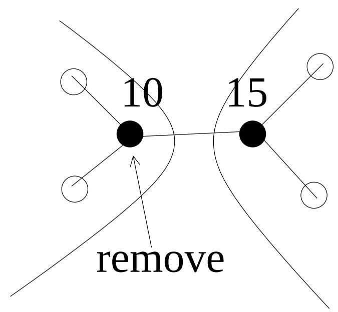
The algorithm works in $O(\log^* n)$, for arbitrary initial weight of edges.

How to $(1-\epsilon)$-aproximate MWIS

We have a planar graph G with weights on vertices $\omega : V(G) \to R^+$

1. Define $\omega(\{u,v\}) :=$ min$(\omega(u), \omega(v))$
2. Compute clusters with weight of connectors $< \epsilon\ \omega(E(G))$
3. In each cluster compute (optimal) MWIS
4. In case of errors on connectors: remove from the solution the vertex of smaller weight.

We know that ω(MWIS$(G)) > \frac{1}{4}\omega(V(G))$ and $\omega(E(G)) < 3\omega(V(G))$ therefore vertices removed in (4) are meaningless ...



How to $(1+\epsilon)$-aproximate MDS

G - planar (unweighted) graph...

1. Find const approximation of MDS D in graph G using *Lenzen, Oswald, Wattenhofer, "What Can Be Approximated Locally?" (SPAA 2008)*

2. For $D = \{v_1, \ldots, v_k\}$ build small clusters $\{W_1, \ldots, W_k\}$ in natural way.

3. Contract each W_i to a single vertex to get graph H, and assing $\omega(e) := 1$ for all $e \in E(H)$.

4. Compute clusters in H s.t. the number of connectors is $< \epsilon\ E(H)$ (also the number of border vertices in clusters is $< O(\epsilon)\ V(H)$).

5. Return to input graph G with big clusters (composed of small clusters) and compute (optimal) MDS D_i in each big clusters C_i.

Why the solution ($\cup_i |D_i|$) is close to optimal?

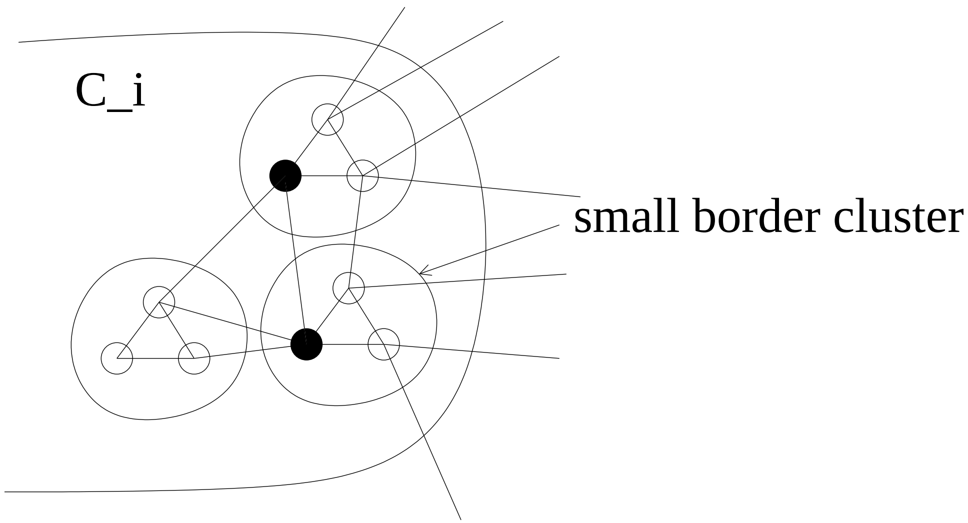
Let $D^* =$ MDS(G), and look at single cluster C_i ...

$$|C_i \cap D^* \cup B_i| \geq |D_i|$$

where B_i is a set of leaders of all small border clusters in C_i

$$|D^*| + \sum_i |B_i| \geq \sum_i |D_i|$$

and $\sum_i |B_i| < \epsilon |V(H)|$, $|V(H)| <$ const $|$MDS(G)$|$.



How to $(1-\epsilon)$-aproximate MIS in constant time with high probability

1. Remove all vertices of degree $> O(1)/\epsilon$

2. In subgraph induced by vertices of degree ≤ 9 compute independent set I in two rounds:

   - each vertex v:

   (a) marks itself with probability 1/2

   (b) unmarks itself if there is a marked neighbour

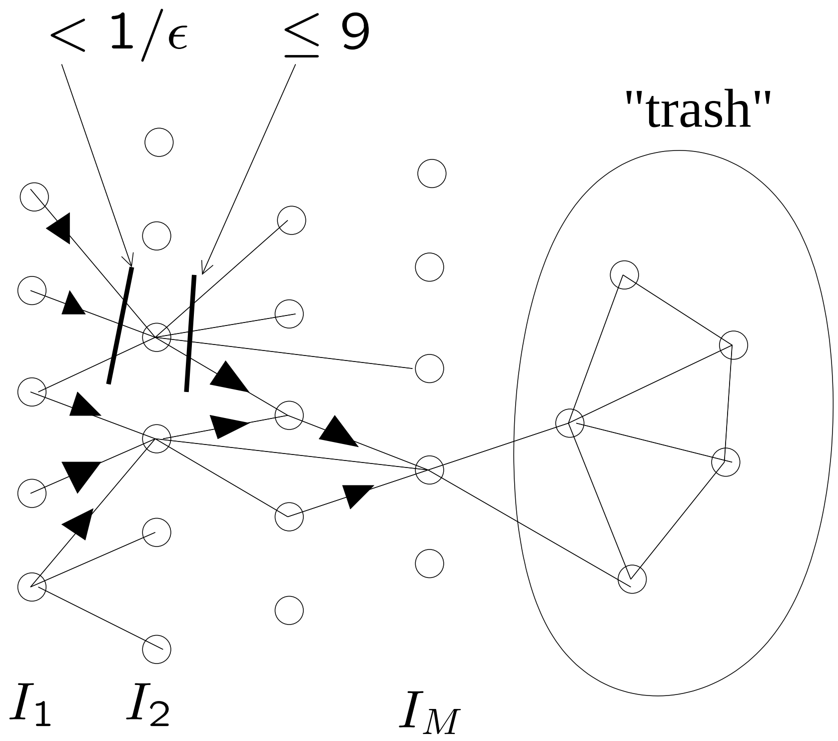
   $|I| > |G|/(2^{9+2}(9^2+1))$ with high probability ...

3. Remove I and repeat that process M times to get sets $I_1, \ldots, I_M$. (M is a constant).

4. Each vertex of I_i puts arrow on the heaviest of its 9 edges to $\cup_{j>i} I_j$. We have a set of rooted trees of diameter $2M$ ...

- We have disjoint subgraphs, with constant number of edges(!), containing constant fraction of $E(G)$, with high probability.

- We can contract them, (re)compute weight of edges, and repeat this process to get clusters and to approximate (unweighted) MIS problem.



Lower bounds

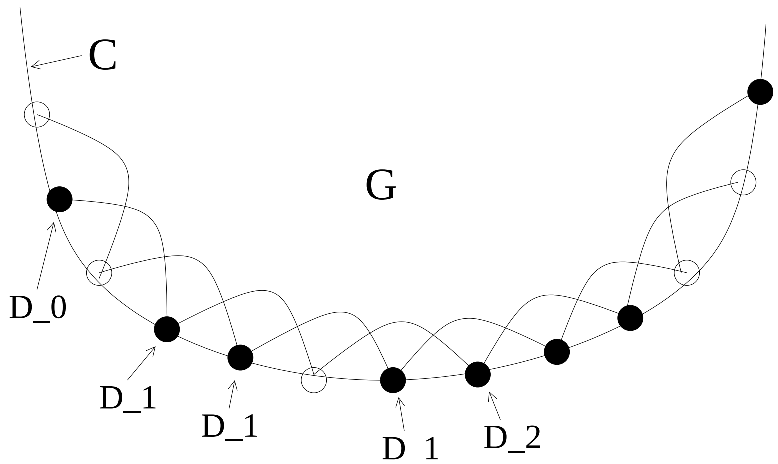
C - a cycle on n vertices.

We can prove that any algorithm working in time T on C can compute independent set of size at most $O(\frac{n}{\log^{(2T)} n})$.

==> const approximation of MIS in C can not be done in constant time

==> it is not possible to compute better than 5-approximation of MDS in planar graphs, in constant time

Proof: Build the following graph G from a cycle C ....



Let D be $(5-\epsilon)$-approx of MDS in G
and lets define $D_i = \{v \in D : deg_{C[D]}(v) = i\}$.

We can construct an independent set I in C in the folowing way:

1. $I := D_0$
2. add to I one of each $D_1 - D_1$ pair
3. add to I all D_1 vertices that have a neighbor from D_2

I is an independent set in C ...

$|\text{MDS}(G)| = \frac{n}{5}$
$4|D_0| + 3|D_1| \geq n - |D|$
$|D| < (5-\epsilon)\text{MDS}(G) = (1 - \frac{\epsilon}{5})n$
$|I| \geq |D_0| + \frac{|D_1|}{2} \geq \frac{\epsilon n}{30}$

I is a const approximation of MIS in C, which can not be approximated in constant time ...